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AN EXAMPLE OF A RIGHT q-RING

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ABSTRACT

We show that Ivanov's classification of indecomposable non-local right q-rings is incomplete and provide a complete classification. Next, we correct and sharpen Byrd's classification of right q-rings.

1. Introduction

Given a ring R, we denote by $\mathrm{Soc}(R)$ and $J(R)$ the right socle and the Jacobson radical of R respectively. Given a nonempty subset S of R, we set

$$r(R;S) = \{\alpha \in R \mid S\alpha = 0\} \quad \text{and} \quad \ell(R;S) = \{\alpha \in R \mid \alpha S = 0\}.$$

Recall that a ring R is said to be a right q-ring provided that all its right ideals are quasi-injective right modules. The study of right q-rings was initiated by Jain, Mohamed and Singh [9] in 1969. In particular they proved the following result which we shall need in the sequel.

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THEOREM 1.1 ([9, Theorem 2.3]): *Let R be a ring. Then R is a right q-ring if and only if R is a right self-injective ring and every essential right ideal of R is an ideal.*

Since then right q-rings have been studied in a number of papers [3, 4, 5, 6, 8, 10, 11, 12, 13]. The reader is referred to [7] for a survey on the subject.

Let n be an integer with $n > 1$, let $D, D_1, D_2, \ldots, D_n$ be skew fields and let V_{ij} be a D_i-D_j-bimodule such that $\dim({}_{D_i}V_{ij}) = 1 = \dim(\{V_{ij}\}_{D_j})$, $1 \le i, j \le n$. Denote by $M_n(D_1, \ldots, D_n; V_{12}, \ldots, V_{n-1,n}, V_{n1})$ the ring of $n \times n$ matrices of the form

$$\begin{pmatrix} D_1 & V_{12} & & & & \\ & D_2 & V_{23} & & & \\ & & D_3 & V_{34} & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & \cdot & D_{n-1} & V_{n-1,n} \\ V_{n1} & & & & & D_n \end{pmatrix}$$

where it is understood that $V_{ij}V_{pq} = 0$ for all i, j, p, q. Next, let α be automorphisms of the skew field D. We denote by V the D-D-bimodule D and by $V(\alpha)$ the D-D-bimodule which as a left D-module is equal to ${}_DD$ and the right D-module structure is given by $x * y = x\alpha(y)$ for all $x \in V(\alpha)$. We set

$$H(n; D; \alpha) = M_n(D, \ldots, D; V, \ldots, V, V(\alpha)).$$

In 1972 Ivanov [4, Theorem 3] proved that an indecomposable right q-ring either is simple Artinian, or is isomorphic to a ring $H(n; D; \mathrm{id}_D)$. Recently Jain, López-Permouth and Syed [8, Theorem 3.13] obtained the following result.

THEOREM 1.2: *Let R be an indecomposable non-local right q-ring. Then R is of the form $M_n(D_1, \ldots, D_n; V_{12}, \ldots, V_{n-1,n}, V_{n1})$ or simple Artinian. Conversely, every ring of the form $M_n(D_1, \ldots, D_n; V_{12}, \ldots, V_{n-1,n}, V_{n1})$ or every simple Artinian ring is a right q-ring.*

We are now in a position to state our first main result.

THEOREM 1.3:

(1) *Every ring $M_n(D_1, \ldots, D_n; V_{12}, \ldots, V_{n-1,n}, V_{n1})$ is isomorphic to a ring of the form $H(n; D; \alpha)$.*

(2) *Two rings $H(n; D; \alpha)$ and $H(n; D; \beta)$ are isomorphic if and only if there exists an automorphism γ of the skew field D such that $\gamma^{-1}\beta\gamma\alpha^{-1}$ is an inner automorphism of D.*

Example: Let C be the complex number field and let α be the complex conjugation. Clearly α is not an inner automorphism of C and so $H(2;C;\alpha) \not\cong H(2;C;\mathrm{id})$ by Theorem 1.3.

The Example shows that the Ivanov's description of an indecomposable non-local right q-rings is not complete. It has to be noted that he made a mistake at the very end of the proof (he wrongly stated that any two D-D-bimodules are isomorphic) and so his proof can be easily corrected.

Let n be a positive integer, let Δ be a right q-ring with essential maximal right ideal P such that the module Δ/P is injective, and clearly it cannot be embedded in ${}_\Delta\Delta$. Since P is an essential right ideal of Δ, it is a two sided ideal by Theorem 1.1 and so $D = \Delta/P$ is a skew field. Next, let $V = {}_DD_D$ be a D-D-bimodule. Clearly V is also a D-Δ-bimodule. We denote by $G(n;\Delta;P)$ the ring of $(n+1)\times(n+1)$ matrices of the form

$$\begin{pmatrix} D & V & & & & \\ & D & V & & & \\ & & D & V & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & \cdot & D & V \\ & & & & & \Delta \end{pmatrix}$$

where it is understood that $VV = 0$. Note that $G(0;\Delta;P) = \Delta$.

Ivanov [4] conjectured that every right q-ring is the direct sum of a finite number of indecomposable non-local right q-rings and a right q-ring all of whose idempotents are central. Byrd [1] classified right q-rings and showed that the structure of right q-rings is more complicated than it was conjectured by Ivanov. We state his main result [1, Theorem 6] in the following slightly different but equivalent form.

THEOREM 1.4: *A right q-ring is isomorphic to a finite direct product of right q-rings of the following kinds:*

(1) *Semisimple Artinian ring.*
(2) $H(n;D;\mathrm{id}_D)$, *where* id_D *is the identity automorphism of* D.
(3) $G(n;\Delta;P)$, *where* Δ *is a right q-ring all of whose idempotents are central.*
(4) *A right q-ring all of whose idempotents are central.*

In view of the Example, Byrd's classification of right q-rings is not complete (see Theorem 1.4(2)). It has to be noted that he made the same mistake as Ivanov in [1, Theorem 3] and his proof can be easily corrected. The main goal of the present paper is to correct and sharpen his classification.

THEOREM 1.5: *A right q-ring R is isomorphic to a finite direct product of right q-rings of the following kinds:*

(1) *Semisimple Artinian ring.*

(2) $H(n;D;\alpha)$, *where α is an automorphism of D.*

(3) $G(n;\Delta;P)$, *where Δ is a strongly regular right self injective ring.*

(4) *A right q-ring all of whose idempotents are central.*

Moreover, R is a left q-ring if and only if it is left self injective.

Note that in contrast with Byrd's Theorem 1.4, the structure of rings in Theorem 1.5(3) is completely described.

2. Proof of main results

Proof of Theorem 1.3: Given

$$d_1 \in D_1, \ldots, d_n \in D_n,\ v_1 \in V_{12}, \ldots, v_{n-1} \in V_{n-1,n}, v_n \in V_{n1},$$

we denote by $[d_1, \ldots, d_n, v_1, \ldots, v_n]$ the matrix

$$\begin{pmatrix} d_1 & v_1 & & & & \\ & d_2 & v_2 & & & \\ & & d_3 & v_3 & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & \cdot & d_{n-1} & v_{n-1} \\ v_n & & & & & d_n \end{pmatrix} \in M_n(D_1, \ldots, D_n; V_{12}, \ldots, V_{n-1,n}, V_{n1}).$$

Clearly

$$\begin{aligned}(1)\qquad &[d_1, \ldots, d_n, v_1, \ldots, v_n][d_1', \ldots, d_n', v_1', \ldots, v_n'] = \\ &[d_1d_1', \ldots, d_nd_n', d_1v_1' + v_1d_2', \ldots, d_{n-1}v_{n-1}' + v_{n-1}d_n', v_nd_1' + d_nv_n']\end{aligned}$$

We are now ready to prove the first statement of the theorem. We set $R = M_n(D_1, \ldots, D_n; V_{12}, \ldots, V_{n-1,n}, V_{n1})$ and $D = D_1$. Choose $0 \neq v \in V_{12}$. Since $\dim({}_DV_{12}) = 1 = \dim(\{V_{12}\}_{D_2})$, $Dv = vD_2$ and so for any $d \in D$ there exists uniquely determined element $\gamma(d) \in D_2$ with $dv = v\gamma(d)$. One may easily check that $\gamma\colon D \to D_2$ is an isomorphism of rings. Clearly V_{23} is a D-D_3-bimodule where the left D-module structure is given via $\gamma\colon D \to D_2$. Let $V = {}_DD_D$ be a D-D-bimodule. We now define a map

$$\beta\colon M_n(D, D, D_3 \ldots, D_n; V, V_{23} \ldots, V_{n-1,n}, V_{n1}) \to R$$

by the rule

$$\beta([d_1,\dots,d_n,v_1,v_2\dots,v_n]) = [d_1,\gamma(d_2),d_3,\dots,d_n,v_1v,v_2,\dots,v_n]$$

(note that v_1v is defined because $v \in V = {}_DD_D$). Clearly β is a bijective additive map. Next, it follows from (1) that

$$\begin{aligned}
&\beta([d_1,\dots,d_n,v_1,\dots,v_n][d_1',\dots,d_n',v_1',\dots,v_n'])\\
&\quad= \beta([d_1d_1',\dots,d_nd_n',d_1v_1'+v_1d_2',\gamma(d_2)v_2'+v_2d_3',\dots,v_nd_1'+d_nv_n'])\\
&\quad= [d_1d_1',\gamma(d_2d_2'),d_3d_3',\dots,(d_1v_1'+v_1d_2')v,\gamma(d_2)v_2'+v_2d_3',\dots,v_nd_1'+d_nv_n']\\
&\quad= [d_1d_1',\gamma(d_2)\gamma(d_2'),d_3d_3',\dots,d_1v_1'v+v_1v\gamma(d_2'),\gamma(d_2)v_2'+v_2d_3',\dots]\\
&\quad= [d_1,\gamma(d_2),d_3,\dots,d_n,v_1v,v_2,\dots,v_n][d_1',\gamma(d_2'),d_3',\dots,d_n',v_1'v,v_2'\dots,v_n']\\
&\quad= \beta([d_1,\dots,d_n,v_1,\dots,v_n])\beta([d_1',\dots,d_n',v_1',\dots,v_n'])
\end{aligned}$$

and so β is an isomorphism of rings. Continuing in this fashion we get that $R \cong M_n(D,\dots,D;V,\dots,V,V_{n1})$.

Fix $0 \neq w \in V_{n1}$. Arguing as above we see that there exists an automorphism α of the skew field D such that $dw = w\alpha(d)$ for all $d \in D$. We now define a map ω: $H(n;D;\alpha^{-1}) \to M_n(D,\dots,D;V,\dots,V,V_{n1})$ by the rule

$$\omega([d_1,\dots,d_n,v_1,v_2\dots,v_n]) = [d_1,d_2,d_3,\dots,d_n,v_1,\dots,v_{n-1},v_nw].$$

Obviously ω is a bijective additive map. Again making use of (1) we get

$$\begin{aligned}
&\omega([d_1,\dots,d_n,v_1,\dots,v_n][d_1',\dots,d_n',v_1',\dots,v_n'])\\
&\quad= \omega([d_1d_1',\dots,d_1v_1'+v_1d_2',\dots,d_{n-1}v_{n-1}'+v_{n-1}d_n',v_n\alpha^{-1}(d_1')+d_nv_n'])\\
&\quad= [d_1d_1',\dots,d_1v_1'+v_1d_2',\dots,d_{n-1}v_{n-1}'+v_{n-1}d_{n-1}',\{v_n\alpha^{-1}(d_1')+d_nv_n'\}w]\\
&\quad= [d_1d_1',\dots,d_1v_1'+v_1d_2',\dots,d_{n-1}v_{n-1}'+v_{n-1}d_{n-1}',v_nwd_1'+d_nv_n'w]\\
&\quad= [d_1,\dots,d_n,v_1,v_2,\dots,v_nw][d_1',\dots,d_n',v_1',v_2'\dots,v_n'w]\\
&\quad= \omega([d_1,\dots,d_n,v_1,\dots,v_n])\omega([d_1',\dots,d_n',v_1',\dots,v_n'])
\end{aligned}$$

which shows that ω is an isomorphism of rings. Thus $R \cong H(n;D;\alpha^{-1})$ and the first statement of Theorem 1.3 is proved.

We shall now prove the second statement of the theorem. We now assume that α and β are isomorphisms of the skew field D such that $H(n;D;\alpha) \cong H(n;D;\beta)$. Let f: $H(n;D;\alpha) \to H(n;D;\beta)$ be an isomorphism of rings.

Given a positive integer i with $1 \leq i \leq n$, we denote by e_i (respectively, e_i') the matrix in $S = H(n;D;\alpha)$ (respectively, $S' = H(n;D;\beta)$) whose (i,i) entry is 1 and all other entries zero. We set $D_i = e_iSe_i$ and $D_i' = e_i'S'e_i'$, $i =$

$1, 2, \ldots, n$. Further, we put $L_i = e_iSe_{i+1}$ and $L'_i = e'_iS'e'_{i+1}$, $i = 1, 2, \ldots, n-1$, and $L_n = e_nSe_1$, $L'_n = e'_nS'e'_1$. Clearly both D_i and D'_i are skew fields isomorphic to D for all $i = 1, 2, \ldots, n$. One can easily check that each L_i (respectively, L'_i) is an ideal of S (respectively, S') which is simple as both left and right S-module (respectively, S'-module). We see that $L_1, L_2, \ldots, L_n$ (respectively, $L'_1, L'_2, \ldots, L'_n$) are all homogeneous components of the socle of S (respectively, S'). Therefore there exists a permutation σ of the set $\{1, 2, \ldots, n\}$ such that $f(L_i) = L'_{\sigma(i)}$ for all $i = 1, 2, \ldots, n$.

Set $L = \sum_{i=1}^n L_i$ and $L' = \sum_{i=1}^n L'_i$ and note that $\mathrm{Soc}(S) = L$, $\mathrm{Soc}(S') = L'$ and

$$f(L) = L'. \tag{2}$$

Set $T = \sum_{i=1}^n D_i \subseteq S$ and $T' = \sum_{i=1}^n D'_i \subseteq S'$. Clearly both T and T' are subrings of S and S' respectively. Moreover, $T = \bigoplus_{i=1}^n D_i$ and $T' = \bigoplus_{i=1}^n D'_i$. Obviously $S = T \oplus L$ and $S' = T' \oplus L'$. Let $\pi\colon S \to T$ and $\pi'\colon S' \to T'$ be canonical projections of abelian groups. Clearly both π and π' are homomorphisms of ring. Since $\ker(\pi) = L$ and $\ker(\pi') = L'$, (2) implies that f induces an isomorphism $g\colon T \to T'$ of rings such that

$$g\pi = \pi' f. \tag{3}$$

Further, since both $L^2 = 0$ and $(L')^2 = 0$,

$$s\ell = \pi(s)\ell, \quad \ell s = \ell\pi(s), \quad s'\ell' = \pi'(s')\ell' \quad \text{and} \quad \ell's' = \ell'\pi'(s') \tag{4}$$

for all $s \in S$, $\ell \in L$, $s' \in S'$ and $\ell' \in L'$. It now follows from both (3) and (4) that

$$f(t\ell) = f(t)f(\ell) = (\pi' f)(t)f(\ell) = (g\pi)(t)f(\ell) = g(t)f(\ell)$$

for all $t \in T$ and $\ell \in L$. We see that

$$f(t\ell) = g(t)f(\ell) \quad \text{and} \quad f(\ell t) = f(\ell)g(t) \quad \text{for all } t \in T \text{ and } \ell \in L. \tag{5}$$

Since the ring T (respectively, T') is the direct sum of skew fields D_i (respectively, D'_i), $i = 1, 2, \ldots, n$, there exists a permutation τ of the set $\{1, 2, \ldots, n\}$ such that $g(D_i) = D'_{\tau(i)}$ for all $i = 1, 2, \ldots, n$. We claim that $\tau = \sigma$. Indeed, assume that $\tau(i) \neq \sigma(i)$ for some $1 \leq i \leq n$. Take $0 \neq t \in D_i$ and $0 \neq \ell \in L_i$. Then $t\ell \neq 0$ and so (5) implies that

$$0 \neq f(t\ell) = g(t)f(\ell) \in D'_{\tau(i)}L'_{\sigma(i)} = 0,$$

a contradiction. Therefore $\tau = \sigma$ and so

$$g(D_i) = D'_{\sigma(i)}, \quad i = 1, 2, \ldots, n. \tag{6}$$

Let $1 \le i \le n$ and $d \in D$. We denote by $d^{(i)}$ the $2n$-tuple $[0, \ldots, 0, d, 0, \ldots, 0]$ where d is on ith position. Analogously, given $1 \le i \le n-1$ and $v \in V$, we denote by $v^{(n+i)}$ the $2n$-tuple $[0, \ldots, 0, v, 0, \ldots, 0]$ where v is on $(n+i)$th position. Finally, given $v \in V(\alpha)$, $v^{(2n)} = [0, \ldots, 0, v]$. Clearly

$$\begin{aligned} d^{(i)}v^{(n+i)} &= (dv)^{(n+i)} \quad \text{for all } i = 1, 2, \ldots, n; \\ v^{(n+i)}d^{(i+1)} &= (vd)^{(n+i)} \quad \text{for all } i = 1, 2, \ldots, n-1; \\ v^{(2n)}d^{(1)} &= (v\alpha(d))^{(2n)}. \end{aligned} \tag{7}$$

Given $1 \le i \le n$, in view of (6), g induces an automorphism g_i of the skew field D such that

$$g(d^{(i)}) = \{g_i(d)\}^{(\sigma(i))} \quad \text{for all } d \in D. \tag{8}$$

Analogously, f induces an automorphism f_i of the additive group of D such that

$$f(v^{(n+i)}) = \{f_i(v)\}^{(n+\sigma(i))} \quad \text{for all } v \in D. \tag{9}$$

Given $d, v \in D$, we claim that

$$\begin{aligned} f_i(dv) &= g_i(d)f_i(v) \quad \text{for all } i = 1, 2, \ldots, n; \\ f_i(vd) &= \begin{cases} f_i(v)g_{i+1}(d) & \text{if } i \ne n \text{ and } \sigma(i) \ne n; \\ f_n(v)(g_1\alpha^{-1})(d) & \text{if } i = n \text{ and } \sigma(n) \ne n; \\ f_i(v)(\beta g_{i+1})(d) & \text{if } i \ne n \text{ and } \sigma(i) = n; \\ f_n(v)(\beta g_1\alpha^{-1})(d) & \text{if } i = n \text{ and } \sigma(n) = n. \end{cases} \end{aligned} \tag{10}$$

Indeed, it follows from (5), (7), (8) and (9) together that

$$\begin{aligned} \{f_i(dv)\}^{(n+\sigma(i))} &= f(\{dv\}^{(n+i)}) = f(d^{(i)}v^{(n+i)}) = g(d^{(i)})f(v^{(n+i)}) \\ &= \{g_i(d)\}^{(\sigma(i))}\{f_i(v)\}^{(n+\sigma(i))} = \{g_i(d)f_i(v)\}^{(n+\sigma(i))} \end{aligned}$$

which proves the first equality. The first case of the second equality in (10) is proved analogously. Now assume that $i = n$ and $\sigma(n) \ne n$. We have

$$\begin{aligned} \{f_n(vd)\}^{(n+\sigma(n))} &= f(\{vd\}^{(2n)}) = f(v^{(2n)}\{\alpha^{-1}(d)\}^{(1)}) \\ &= f(v^{(2n)})g(\{\alpha^{-1}(d)\}^{(1)}) \\ &= \{f_n(v)\}^{(n+\sigma(n))}\{(g_1\alpha^{-1})(d)\}^{(\sigma(1))} \\ &= \{f_n(v)(g_1\alpha^{-1})(d)\}^{(n+\sigma(n))} \end{aligned}$$

(note that $\sigma(n) < n$ forces $\sigma(1) = \sigma(n) + 1$). The last two cases of the second equality in (10) are proved analogously.

Setting $v = 1$ and $t_i = g_i(1)$, $i = 1, 2, \ldots, n$, we get from (10) that

$$f_i(d) = g_i(d)t_i \quad \text{for all } i = 1, 2, \ldots, n;$$
$$f_i(d) = \begin{cases} t_i g_{i+1}(d) & \text{if } i \neq n \text{ and } \sigma(i) \neq n; \\ t_n(g_1\alpha^{-1})(d) & \text{if } i = n \text{ and } \sigma(n) \neq n; \\ t_i(\beta g_{i+1})(d) & \text{if } i \neq n \text{ and } \sigma(i) = n; \\ t_n(\beta g_1 \alpha^{-1})(d) & \text{if } i = n \text{ and } \sigma(n) = n \end{cases}$$

for all $d \in D$ and so

$$(11) \qquad g_i(d)t_i = \begin{cases} t_i g_{i+1}(d) & \text{if } i \neq n \text{ and } \sigma(i) \neq n; \\ t_n(g_1\alpha^{-1})(d) & \text{if } i = n \text{ and } \sigma(n) \neq n; \\ t_i(\beta g_{i+1})(d) & \text{if } i \neq n \text{ and } \sigma(i) = n; \\ t_n(\beta g_1 \alpha^{-1})(d) & \text{if } i = n \text{ and } \sigma(n) = n. \end{cases}$$

Let G be the group of all automorphisms of the skew field D and let N be the subgroup of all inner automorphisms of D. It is well-known that N is a normal subgroup of G. Let $g \mapsto \overline{g}$ be the canonical projection of groups $G \to G/N$. Note that each $g_i \in G$.

There are two cases to consider.

CASE 1: Suppose that $\sigma(j) = n$ for some $1 \leq j < n$. It follows from (11) that

$$\overline{g_1} = \overline{g_2} = \cdots = \overline{g_j} = \overline{\beta g_{j+1}} \quad \text{and} \quad g_{j+1} = \cdots = \overline{g_{n-1}} = \overline{g_n} = \overline{g_1\alpha^{-1}}.$$

and so $\overline{\beta g_{j+1}} = \overline{g_1} = \overline{g_{j+1}\alpha}$ forcing $\gamma^{-1}\beta\gamma\alpha^{-1} \in N$ with $\gamma = g_{j+1}$.

CASE 2: Suppose that $\sigma(n) = n$. According to (11) we have that

$$\overline{g_1} = \overline{g_2} = \cdots = \overline{g_{n-1}} = \overline{g_n} = \overline{\beta g_1 \alpha^{-1}}$$

and so $\overline{g_1} = \overline{\beta g_1 \alpha^{-1}}$ forcing $\gamma^{-1}\beta\gamma\alpha^{-1} \in N$ with $\gamma = g_1$.

To complete the proof of the theorem, assume that $\gamma^{-1}\beta\gamma\alpha^{-1} \in N$ for some $\gamma \in G$. Set $\delta = \gamma^{-1}\beta\gamma$. Define a map f: $H(n; D; \beta) \to H(n; D; \delta)$ by the rule

$$f([d_1, \ldots, d_n, v_1, \ldots, v_n]) = [\gamma^{-1}(d_1), \ldots, \gamma^{-1}(d_n), \gamma^{-1}(v_1), \ldots, \gamma^{-1}(v_n)].$$

Clearly f is a bijective additive map. It follows from (1) directly that f is an isomorphism of rings. Therefore it is enough to show that $H(n; D; \delta) \cong H(n; D; \alpha)$. Choose a nonzero element $t \in D$ such that $\delta(d) = t\alpha(d)t^{-1}$ for all $d \in D$. Clearly $\delta(d)t = t\alpha(d)$, $d \in D$. Define a map g: $H(n; D; \delta) \to H(n; D; \alpha)$ by the rule

$$g([d_1, \ldots, d_n, v_1, \ldots, v_n]) = [d_1, \ldots, d_n, v_1, \ldots, v_{n-1}, v_n t].$$

Obviously g is a bijective additive map. It now follows from (1) that

$$\begin{aligned}
&g([d_1,\dots,d_n,v_1,\dots,v_n][d'_1,\dots,d'_n,v'_1,\dots,v'_n])\\
=&g([d_1d'_1,\dots,d_nd'_n,d_1v'_1+v_1d'_2,\dots,d_{n-1}v'_{n-1}+v_{n-1}d'_n,v_n\delta(d'_1)+d_nv'_n])\\
=&[d_1d'_1,\dots,d_nd'_n,d_1v'_1+v_1d'_2,\dots,d_{n-1}v'_{n-1}+v_{n-1}d'_n,(v_n\delta(d'_1)+d_nv'_n)t]\\
=&[d_1d'_1,\dots,d_nd'_n,d_1v'_1+v_1d'_2,\dots,d_{n-1}v'_{n-1}+v_{n-1}d'_n,v_nt\alpha(d'_1)+d_nv'_nt]\\
=&[d_1,\dots,d_n,v_1,\dots,v_nt][d'_1,\dots,d'_n,v'_1,\dots,v'_nt]\\
=&g([d_1,\dots,d_n,v_1,\dots,v_n])g([d'_1,\dots,d'_n,v'_1,\dots,v'_n])
\end{aligned}$$

which shows that g is an isomorphism of rings. The proof is thereby complete. ∎

LEMMA 2.1: *Let A be a right self injective ring in which every idempotent is central and let M be an injective simple right A-module. Then there exists an idempotent $v \in A$ such that $Mv = M$ and vA is a strongly regular right self injective ring.*

Proof: Let $J(A)$ denotes the Jacobson radical of A. Set $\overline{A} = A/J(A)$ and let $a \mapsto \bar{a}$, $a \in A$, be the canonical projection of rings $A \to \overline{A}$. It follows from [15, Corollary 4.10 and Theorem 4.7] that every idempotent of $\overline{A}$ is of the form $\overline{u}$ where $u = u^2 \in A$. Therefore idempotents in $\overline{A}$ are central and

(12) $\overline{A}$ is a right self injective strongly regular ring.

It follows from Zorn's lemma that there exists a family $\{x_i \mid i \in I\}$ of elements of $J(A)$ maximal with respect to the property $\sum_{i\in I} x_iA = \bigoplus_{i\in I} x_iA$. Set $K = \bigoplus_{i\in I} x_iA$. Let wA be an injective hull of K where $w = w^2 \in A$. Pick idempotents $w_i \in wA$, $i \in I$, such that each w_iA is an injective hull of x_iA. Since idempotents in A are central and $x_i \in w_iA$, we have

(13) $ww_i = w_i = w_iw$ and $w_ix_i = x_i = x_iw_i$ for all $i \in I$.

As $\sum_{i\in I} x_iA = \bigoplus_{i\in I} x_iA$, we conclude that $\sum_{i\in I} w_iA = \bigoplus_{i\in I} w_iA$ and so

(14) $\{w_i \mid i \in I\}$ is a family of pairwise orthogonal central idempotents of A.

Obviously

(15) wA is an injective hull of $\bigoplus_{i\in I} w_iA$.

We claim that

$$wJ(A) = J(A). \tag{16}$$

It is enough to show that $(1-w)J(A) = 0$. Assume to the contrary that $(1-w)J(A) \neq 0$ and pick $0 \neq y \in (1-w)J(A)$. Clearly $yA \cap wA = 0$ and whence $yA \cap \bigoplus_{i\in I} x_iA = 0$, a contradiction to the choice of the family $\{x_i \mid i \in I\}$. Therefore (16) is proved.

We define a homomorphism of right A-modules $f\colon \bigoplus_{i\in I} w_iA \to \bigoplus_{i\in I} x_iA$ by the rule $w_i \mapsto x_iw_i$, $i \in I$. Since w_i is central, we see that $f(w_i) = w_ix_i$, $i \in I$. Recalling that wA is an injective right A-module containing both $\bigoplus_{i\in I} w_iA$ and $\bigoplus_{i\in I} x_iA$, we conclude that f can be extended to an endomorphism of wA_A. Obviously $\mathrm{End}(wA_A) = wAw$ and so there exists $x \in wAw$ such that $f(y) = xy$ for all $y \in \bigoplus_{i\in I} w_iA$. In particular, $w_ix_i = f(w_i) = xw_i = w_ix$ and because $x_iA \subseteq w_iA$, we have that

$$w_ix = w_ix_i = x_i \quad \text{for all } i \in I. \tag{17}$$

We claim that

$$x \in J(A), \quad \text{and} \quad wA \text{ is an injective hull of } xA. \tag{18}$$

Indeed, assume to the contrary that $x \notin J(A)$. Clearly $\overline{x}$ is a nonzero element of the ring $\overline{A}$ and so (12) implies that there exists $y \in A$ such that $\overline{xy}$ is a nonzero idempotent. It is well-known that there exists a nonzero idempotent $u \in A$ with $xy - u \in J(A)$ (see [15, Corollary 4.10 and Theorem 4.7]). Since $x \in wA$, $(1-w)x = 0$ and so $(1-w)u \in J(A)$. Recalling that idempotents in A are central, we see that $(1-w)u = 0$ because the Jacobson radical of a ring does not contain nonzero idempotents. Therefore $wu = u$ and whence $0 \neq uA \subseteq wA$. Next, $J(A) \ni w_i(xy-u) = (w_ix)y - w_iu = x_iy - w_iu$ by (17) and so $w_iu \in J(A)$ (because $x_i \in J(A)$) forcing $w_iu = 0$ for all $i \in I$. Since $\bigoplus_{i\in I} x_iA$ is an essential submodule of wA and $0 \neq uA \subseteq wA$, there exist a nonempty finite subset $I_0 \subseteq I$ and elements $a_i \in A$, $i \in I_0$, such that $0 \neq \sum_{i\in I_0} x_ia_i \in uA \cap \bigoplus_{i\in I} x_iA$. Pick $j \in I_0$ with $x_ja_j \neq 0$. Then $0 \neq x_ja_j = w_j \sum_{i\in I_0} x_ia_i \in w_juA = 0$, a contradiction. Therefore $x \in J(A)$. Finally, by (17), $xA \supseteq xw_iA = w_ixA = x_iA$ for all $i \in I$ and so $xA \supseteq K$. We see that xA is an essential submodule of wA and whence (18) is proved.

Now let $P = r(A; M)$. Since M is a simple right A-module, P is a primitive right ideal of A. Let $B = \{b \in A \mid b^2 = b \in P\}$ and let $Q = \sum_{b\in B} bA$. Clearly $Q \subseteq P$. We claim that

$$P = J(A) + Q. \tag{19}$$

Indeed, the inclusion $P \supseteq J(A) + Q$ is obvious. Therefore it is enough to show that $\overline{P} \subseteq \overline{Q}$. Every ideal of a regular ring is generated by idempotents. Given an idempotent $\overline{u} \in \overline{P}$, $u \in P$, by [15, Corollary 4.10 and Theorem 4.7] there exists an idempotent $a \in A$ with $u - a \in J(A) \subseteq P$ (i.e., $\overline{a} = \overline{u}$) and so $a \in P$ forcing $a \in B$ and whence (19) is proved.

Setting $A' = A/Q$, we denote by $a \mapsto a'$ the canonical projection of rings $A \to A'$. Obviously $P' = J(A)'$ forcing $P' \subseteq J(A')$. Clearly $A'/P' \cong A/P \cong \overline{A}/\overline{P}$. We see that $\overline{A}/\overline{P}$ is a skew field because $\overline{P}$ is a primitive right ideal of the strongly regular ring $\overline{A}$ (see (12)). Therefore A'/P' is a skew field as well. We conclude that

(20) A' is a local ring with maximal ideal $P' = J(A')$.

Since $Q \subseteq P = r(A; M)$, M is A'-module canonically. By assumption M is an injective simple right A-module. Therefore M is an injective simple right A'-module. Since over a local ring simple modules are isomorphic, we conclude from (20) that A' is a right V-ring. Therefore $J(A') = 0$ by [16, 23.1(i)] forcing $P = Q$. By (18), $x \in J(A) \subseteq P = Q$ and whence there exist $b_1, b_2, \ldots, b_n \in B$ and $a_1, a_2, \ldots, a_n \in A$ such that $x = \sum_{i=1}^{n} b_i a_i$. Since each $b_i \in P$, $1 - b_i \notin P$. Recalling that P is a prime ideal of A and idempotents in A are central, we conclude that $v = (1 - b_1)(1 - b_2)\ldots(1 - b_n) \notin P$. Therefore $1 - v \in P$ forcing $M(1 - v) = 0$ and so $Mv = M$. Next, clearly $vb_t = 0$ for all $t = 1, 2, \ldots, n$ and whence $vx = \sum_{i=1}^{n} vb_i a_i = 0$. Therefore $vA \cap xA = 0$. It now follows from (18) that $vA \cap wA = 0$ and whence (16) implies that $vA \cap J(A) = 0$. We see that $vA \cong (vA)/(vA \cap J(A)) = \overline{v}\overline{A}$ is a right self injective strongly regular ring because $\overline{A}$ is so by (12). The proof is complete. ∎

Remark 2.2: A ring $R = G(n; \Delta; P)$ is not left self injective.

Proof: Given an integer $1 \leq i \leq n$, we denote by e_i the matrix in R whose (i,i) entry is 1 and all other entries zero. Set $e = e_1$ and $L = e_1 R e_2$. Clearly $eRe \cong \Delta/P = D$ is a skew field and $\dim({}_{eRe}L) = 1$. Choose any $0 \neq p \in L$ and note that $eRep = L$. Define a map $f\colon L \to eRe$ by the rule $(xp)f = x$ for all $x \in eRe$. Clearly f is a well-defined additive map. We claim that f is a homomorphism of left R-modules. To this end, note that $Re = eRe$. Take $y \in L$ and $z \in R$. Clearly $y = xp$ for some $x \in eRe$. Since $ey = y$, $zy = (ze)y$. Recalling that $Re = eRe$, we see that $ze = eze$. Therefore $zy = (ezexe)p$ and so

$$\begin{aligned}(zy)f &= ([ezexe]p)f = ezexe = (eze)(exe) = (ze)(exe)\\ &= z(exe) = z[(exep)f] = z(yf)\end{aligned}$$

which proves our claim. Since ${}_RR$ is injective, there exists $r \in R$ such that $yf = yr$ for all $y \in L$. As $ye_2 = y$ for all $y \in L$, we may assume without loss of generality that $r \in e_2R$. As $e_2R = e_2Re_2 + e_2Re_3$, we see that $re = re_1 = 0$, contradicting $Lf \in eRe$. The proof is thereby complete. ∎

Remark 2.3: Let A be a ring such that the following conditions are fulfilled:

(1) Every right ideal of A is two sided.

(2) For any $a \in A$ and any A-module map $f: aA \to A$ there exists an element $b \in A$ such that $fx = xb$ for all $x \in aA$.

Then every left ideal of A is two sided.

Proof: It is enough to show that Aa is an ideal of A for any $a \in A$. Given a nonempty subset $S \subseteq A$, we set

$$\ell(S) = \{x \in A \mid xS = 0\} \quad \text{and} \quad r(S) = \{x \in A \mid Sx = 0\}.$$

We claim that $\ell(r(Aa)) = Aa$. Indeed, $Aa \subseteq \ell(r(Aa))$. Let $b \in \ell(r(Aa))$. Then $bx = 0$ for all $x \in A$ with $ax = 0$. Therefore the map $f: aA \to bA$, $ax \mapsto bx$, is well defined and so there exists an element $c \in A$ with $fy = cy$ for all $y \in Aa$. In particular, $b = fa = ac$ which prooves our claim. Finally, $r(Aa)$ is a right ideal and so it is an ideal of A. Therefore $Aa = \ell(r(Aa))$ is an ideal of A as well. The proof is completed. ∎

Incidentally, an immediate consequence of the above remark is that every right duo right principally injective ring is left duo.

Proof of Theorem 1.5: Making use of Theorems 1.2 and 1.3, we obtain that every right q-ring R is isomorphic to a finite direct product of rings of the following kinds:

(1) Semisimple Artinian ring.

(2) $H(n; D; \alpha)$, where α is an automorphism of D.

(3) $G(n; \Delta; P)$, where Δ is a right q-ring all of whose idempotents are central.

(4) A right q-ring all of whose idempotents are central.

Now consider a ring $G(n; \Delta; P)$. Since $M = \Delta/P$ is an injective Δ-module, Lemma 2.1 implies that there exists an idempotent $v \in \Delta$ such that $Mv = M$ and $v\Delta$ is a strongly regular right self injective ring. The first part of Theorem 1.5 now follows from obvious isomorphism $G_(n; \Delta; P) \cong G(n; v\Delta; vP) \times (1 - v)\Delta$ together with Remark 2.3.

To prove the last statement of the theorem ,we now assume that R is left self injective. It follows from both Remark 2.2 and the first part of the theorem that R is isomorphic to a finite direct product of rings of the following kinds:

(1) Semisimple Artinian ring.

(2) $H(n; D; \alpha)$, where α is an automorphism of D.

(3) A right q-ring all of whose idempotents are central.

Clearly every semisimple artinian ring is both a left and a right q-ring. According to [8, Corollary 3.12] every ring of the form $H(n; D; \alpha)$ is a left q-ring. Finally, it follows from both Theorem 1.1 and Remark 2.3 that every self injective right q-ring is also a left q-ring. The proof is thereby complete. ∎

References

[1] K. A. Byrd, *Right self-injective rings whose essential ideals are two sided,* Pacific Journal of Mathematics **82** (1979), 23–41.

[2] C. Faith, *Algebra I; Rings, Modules, and Categories,* Springer-Verlag, Berlin, 1981.

[3] D. A. Hill, *Semi-perfect q-rings,* Mathematische Annalen **200** (1973), 113–121.

[4] G. Ivanov, *Non-local rings whose ideals are quasi-injective,* Bulletin of the Australian Mathematical Society **6** (1972), 45–52.

[5] G. Ivanov, *Addendum to "Non-local rings whose ideals are quasi-injective",* Bulletin of the Australian Mathematical Society **12** (1975), 159–160.

[6] G. Ivanov, *On a generalization of injective von Neumann rings,* Proceedings of the American Mathematical Society **124** (1996), 1051–1060.

[7] S. K. Jain, *Rings whose cyclic modules have certain properties and the duals,* in *Ring Theory*, Vol. 25, Proceedings of the Ohio University Conference, 1976, Marcel Dekker, 1977.

[8] S. K. Jain, S. R. López-Permouth and S. R. Syed, *Rings with quasi-continuous right ideals,* Glasgow Mathematical Journal **41** (1999), 167–181.

[9] S. K. Jain, S. H. Mohamed and S. Singh, *Rings in which every right ideal is quasi-injective,* Pacific Journal of Mathematics **31** (1969), 73–79.

[10] A. Koehler, *Rings for which every cyclic module is quasi-projective,* Mathematische Annalen **189** (1970), 407–419.

[11] A. Koehler, *Rings with qusi-injective cyclic modules,* The Quarterly Journal of Mathematics. Oxford **25** (1974), 51–55.

[12] S. H. Mohamed, *Rings whose homomorphic images are q-rings,* Pacific Journal of Mathematics **35** (1970), 727–735.

[13] S. H. Mohamed, *q-rings with chain conditions,* Journal of the London Mathematical Society **2** (1972), 455–460.

[14] B. J. Müllear and S. Mohamed, *Continuous and Discrete Modules,* Cambridge University Press, 1990.

[15] Y. Utumi, *On continuous rings and self-injective rings*, Transactions of the American Mathematical Society **118** (1965), 158–173.

[16] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, London, 1991.